Quantum computation with abelian anyons

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Abstract: A universal quantum computer can be constructed using abelian anyons. Two qubit quantum logic gates such as controlled-NOT operations are performed using topological effects. Single-anyon operations such as hopping from site to site on a lattice suffice to perform all quantum logic operations. Quantum computation using abelian anyons shares some but not all of the robustness of quantum computation using non-abelian anyons.

A wide variety of methods can be used to construct quantum computers in principle (1-18). Essentially any interaction between two quantum degrees of freedom suffices to construct universal quantum logic gates (10-11). In particular, Kitaev (12) has shown that quantum computation can be effected using non-abelian anyons. The resulting quantum computation is intrinsically fault-tolerant. This paper shows that universal quantum computation can be effected using abelian anyons: two-qubit quantum logic gates, such as the controlled-NOT gate, are enacted topologically via the phase factor of $e^{i\phi}$ that occurs when one anyon is moved around another. Quantum computation using abelian anyons shares some but not all of the robustness of quantum computation using non-abelian anyons. Possible realizations in terms of solid-state systems and interferometers are discussed, and issues of noise and decoherence are investigated.

A quantum logic gate is an operation that transforms quantum-information bearing degrees of freedom. A set of quantum logic gates is universal if arbitrary quantum computations can be built up by repeatedly applying gates from the set to different qubits. To prove that abelian anyons are capable of universal quantum computation, we will show

that single anyon quantum logic gates such as those that move an anyon from one site to another form a universal set: universal quantum computation can be effected simply by moving fermions around a lattice or network.

Consider the case of anyons moving on a two-dimensional lattice. Each site j of the lattice corresponds to a local mode that can be either occupied by an anyon, $|+\rangle_j$, or unoccupied $|-\rangle_j$. For example, the anyons could be quantum-Hall effect excitations on a two-dimensional spatial lattice.

Now consider operations that can be performed on anyons. Let $b_j, b_j^\dagger$ be the annihilation and creation operators for the j th mode. First, applying the Hamiltonian $A_j = b_j^\dagger b_j$ leaves $|-\rangle_j$ unchanged and multiplies the state $|+\rangle_j$ by a phase. Second, if j and k are two adjacent sites or modes, applying the Hamiltonian $B_{jk} = b_j^\dagger b_k + b_k^\dagger b_j$ 'swaps' the states of the two modes. Clearly, the anyons on the lattice can be moved around at will by repeated swapping operations. Finally, when one anyon is moved around another anyon, its state acquires a phase of $e^{i\phi}$. As will now be shown, this is all that is required to effect universal quantum computation.

The trick to performing universal quantum computation using abelian anyons is to store a quantum bit on a single anyon at *two* sites. Associate two sites j, j' with the jth qubit, and define the jth qubit by $|0\rangle_j = |+-\rangle_{jj'}$ and $|1\rangle_j = |-+\rangle_{jj'}$. The operations described above then map in a straightforward way onto the usual quantum logic operations on these qubits. The Hamiltonian $A_j = |1\rangle_j\langle 1| = (\sigma_z^j + 1)/2$ corresponds to a rotation about the z-axis, where $\sigma_z = |0\rangle\langle 0| + |1\rangle\langle 1|$ Swapping j and j' then corresponds to a NOT operation, and a partial swap corresponds to a rotation $e^{-i\theta\sigma_x/2}$, where $\sigma_x = B_{jj'} = |0\rangle\langle 1| + |1\rangle\langle 0|$. Since any single-qubit rotation can be built up out of rotations about x and z axes, the ability to apply A_j and B_{jk} translates into the ability to apply arbitrary single-qubit rotations. Note that although B_{jk} operates on two modes or sites, it involves no direct interactions between anyons, as each of our two-site qubits contains exactly one anyons.

To perform a two-qubit operation on two of our two-site qubits $|x\rangle_j$, $|y\rangle_k$, simply take whatever is in the first site of the jth qubit, and by repeated swaps, move it around the first site of the kth qubit. A convenient way to visualize such an operation is to think of

time as a third dimension, so that moving the contents of one site around another is a braiding action on the time-lines of the sites. The exact path taken does not matter as long as it goes around no other qubit sites that might contain a anyon: the site is braided around one and only one other site. The overall state of the two qubits then acquires a phase of -1 if and only if the first site of the jth qubit and the first site of the kth qubit originally contain anyons. Otherwise, no anyon is moved around another, and the state remains unchanged. That is, we have

$$\begin{aligned} |00\rangle &\rightarrow |00\rangle \\ |01\rangle &\rightarrow |01\rangle \\ |10\rangle &\rightarrow |10\rangle \\ |11\rangle &\rightarrow e^{i\phi}|11\rangle \quad . \end{aligned} \tag{1}$$

But this is just a controlled-phase gate, closely related to a so-called controlled-NOT gate (indeed, for $\phi = \pi$ as in the case of semions, the controlled phase gate can be turned into a controlled-NOT gate by application of σ_x rotations to the second qubit). Single qubit rotations and controlled phase gates together form a universal set of quantum logic gates. This proves our basic result: a set of universal quantum logic gates can be constructed using abelian anyons. Note that all qubits are stored on single anyons, which can be kept an arbitrary distance from each other during the course of the quantum computation.

An arbitrary quantum computation can be enacted as follows. First, map out a quantum circuit diagram for the computation in terms of elementary quantum logic gates. Program an array of two-site qubits with the proper initial states by moving an anyon to the proper site of each qubit. Enact one-qubit gates by phase shifts and partial swaps on the two sites corresponding to the qubit, and enact two-qubit operations by 'braiding' the contents of the first site of the first qubit about the first site of the second qubit. Read out the answer by determining the location of the fermions in the two-site output qubits.

How might one realize such a quantum computer? Clearly, the above discussion suggests that a two-dimensional lattice of abelian anyons. The anyons of the quantum Hall effect will do ($\phi = 2\pi/3$). All that is required is the ability to perform accurate phase-shifts and swaps. These operations are local and act on single anyons. They could

be enacted by applying localized potentials via, e.g., nanofabricated electrodes or scanning tunneling microscopes. Two-qubit operations are topological in nature, and hence are robust to local, anyon-number preserving errors, just as in anyonic quantum computation (12-13). Single qubit operations are not topological in nature and are less robust. If the anyons are massive and the modes in the lattice are spatially separated, then they will be subject to decoherence due to the environment effectively 'detecting' whether or not there is an in a particular site (20-21).

If the anyons have an additional degree of freedom such as a magnetic moment, then a conceptually elegant, though technically difficult way of performing this type of anyonic quantum computation is to use interferometry in two dimensions. Here, rather than storing a qubit on two anyons, one can store it on the spin of a single anyon just as in NMR. Single qubit quantum logic operations can then be enacted by applying magnetic fields as in NMR quantum computation. The topological two-qubit gate can be enacted by applying a magnetic field gradient, as in a Stern-Gerlach apparatus. The gradient field diverts the jth fermion into one of two different modes, depending on whether or its spin is $|-1/2\rangle$ or $|+1/2\rangle$. To perform the topological two-qubit phase shift gate, apply gradients to two qubits j and k, braid the first mode of the jth qubit about the first mode of the kth qubit, then recombine the two modes of the jth qubit and the two modes of the kth qubit using gradient fields. That is, one creates two 'braided' Stern-Gerlach apparatuses by linking two of the four arms. The resulting 'braided' Stern-Gerlach apparatus performs the controlled phase shift. Accordingly, an anyonic quantum computer can in principle be constructed using interferometry alone. (In contrast, when one attempts to construct purely interferometric 'bosonic' quantum computers using photons, quantum computation can only be performed by using exponentially more resources than a conventional quantum computers (22-23).)

Quantum computation with abelian anyons is likely to be quite difficult to accomplish in practice. Abelian anyons, however, though exotic, are less exotic than the non-abelian anyons of Kitaev. The reason that non-abelian effects are not required in this case is that the method of constructing quantum logic gates is not entirely topological: one-qubit gates are performed using 'conventional' quantum logic operations such as hopping. Only the

two-qubit gates are topological in nature. As a result, although abelian anyons represent an experimentally more accessible path to anyonic quantum computation than that provided by non-abelian anyons, the resulting quantum computation is less fault-tolerant. It is to be hoped, however, that the universal quantum computation by abelian anyons will open up new avenues to understanding topological effects in quantum computation (16-17).

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